THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Feb 28

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

Part I: Additional exercises

1. Show that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if every subsequence of (x_n) has in turn a subsequence (sometimes we use the word subsubsequence) that converges to x.

   Proof. " $\Longrightarrow$ ": It is a direct consequence of **Theorem 3.4.2**. Let (x_{n_k}) be any subsequence of (x_n). Then (x_{n_k}) converges to x, which is a subsubsequence of (x_{n_k}) itself.

   " $\Longleftarrow$ ": We will prove by contradiction. Suppose (x_n) does not converges to x. Then refer to **Theorem 3.4.4** and there exists some $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) such that

   $$|x_{n_k} - x| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \qquad (*)$$

   But by assumption, (x_{n_k}) has a subsubsequence $(x_{n_{k_j}})$ which converges to x. Take $\varepsilon = \varepsilon_0 > 0$, then there exists $N \in \mathbb{N}$ such that

   $$|x_{n_{k_j}} - x| < \varepsilon = \varepsilon_0, \quad \forall j \geq N$$

   which contradicts with $(*)$. Therefore, (x_n) must converge to x.

2. Suppose (x_n) is a **monotone** sequence of real numbers and (x_n) has a convergent subsequence, show that (x_n) itself is convergent.

   Proof: WLOG, we assume that (x_n) is an increasing sequence and has a subsequence (x_{n_k}) which converges to $x \in \mathbb{R}$. Then we have that (x_{n_k}) is bounded (also increasing) and

   $$x_{n_k} \leq x = \sup\{x_{n_k}\}, \quad \forall k \in \mathbb{N}.$$

   $\forall k \in \mathbb{N}$, it indicates that $x_k \leq x_{n_k} \leq x$ since $k \leq n_k$ (why?) and (x_n) is increasing. Therefore, (x_n) itself is also bounded above and consequently convergent by MCT.

3. Let (x_n) be a bounded sequence that does not converge to $x \in \mathbb{R}$. Show that there exists a subsequence of (x_n) that converges to some $x' \neq x$.

   Proof: This is a variation of **Theorem 3.4.9**. Please refer to the textbook and also notice that the condition that (x_n) is bounded cannot be dropped.

4. **(Generalizations of Ex 3.4.10)** Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \geq n\}$.

   (a) Show that (s_n) is bounded.

   (b) Show that (s_n) is monotonically decreasing. Hence, by MCT we have that (s_n) converges to $S = \inf\{s_n\}$ (In standard notations we denote it by $\limsup x_n$ or $\overline{\lim_{n\to\infty}} x_n$).

   (c) Show that there is a subsequence of (x_n) that converges to S.

(d) It is easily seen that the assumption that (x_n) is bounded from above cannot be dropped. However, show that the the assumption that (x_n) is bounded from below cannot be dropped either by giving a counterexample.

Solution:

(a)-(b) Refer to Prof. Chou's lecture notes.

(c) Since $S = \inf\{s_n\}$, for any $n \in \mathbb{N}$ we can choose m_n such that $S \le s_{m_n} < S + \frac{1}{n}$. Now by definition $s_{m_n} = \sup\{x_k : k \ge m_n\}$ and thus we can choose $k_n \ge m_n$ such that $s_{m_n} - \frac{1}{n} < x_{k_n} \le s_{m_n}$.
Consider the subsequence (x_{k_n}) and it converges to S because

$$S - \frac{1}{n} \le s_{m_n} - \frac{1}{n} < x_{k_n} \le s_{m_n} < S + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

(d) Consider the sequence $(x_n) = (-1, -2, -3, -4, \cdots)$ which is not bounded from below. Then $s_n = -n$ and (s_n) does not converge to any real number.

Part II: Some comments

1. Suppose (x_n) is a sequence of real numbers defined by $x_n = \frac{n}{n+1} \cos \frac{n\pi}{2}$. We can find $\underline{\lim}_{n\to\infty} x_n$, $\overline{\lim}_{n\to\infty} x_n$ in two ways.

   First, by definition we have (please check the following results yourselves)

   $$s_n = \sup\{x_k : k \ge n\} = 1, \quad i_n = \inf\{x_k : k \ge n\} = -1$$

   and thus

   $$\overline{\lim_{n\to\infty}}\, x_n = \inf\{s_n\} = 1, \quad \underline{\lim_{n\to\infty}}\, x_n = \sup\{i_n\} = -1.$$

   On the other hand, we can find the set of limit points of (x_n) as to be $S = \{-1, 0, 1\}$ and therefore

   $$\overline{\lim_{n\to\infty}}\, x_n = \sup S = 1, \quad \underline{\lim_{n\to\infty}}\, x_n = \inf S = -1.$$

2. In the definition of Cauchy sequence, the indices m, n should be **independent** (however, we can always assume that $m > n$ in applications). They are arbitrary, as long as large enough $(\ge H(\varepsilon))$.

   But when proving that a given sequence is NOT a Cauchy sequence, we are allowed to (and it is usually useful) to specify a relation between m and n.

   We cannot emphasize the importance of Cauchy criteria too much. You will meet it frequently in your later study and you should have a good understanding of Cauchy criteria.

   Remark: The definition of a sequence (x_n) that violates Cauchy criteria is: there exists some $\varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, there exist natural numbers $n_0 > N, m_0 > N$ such that

   $$|x_{n_0} - x_{m_0}| \ge \varepsilon_0.$$

   m_0, n_0 here can have a relation with each other.

3. **(3.4.5) Divergence Criteria** is very useful and usually much more convenient than proving by definition. For example, in Quiz 2b you are required to show the sequence $\left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \cdots\right\}$ is divergent. Now we can find two subsequences that have different limits:

$$(1, 1, 1, \cdots) \to 1, \quad \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right) \to 0 \quad \text{as} \quad n \to \infty.$$

4. **Two common mistakes you made in Assignments 5-6**

   (a) **3.3.10** Some of you use

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \lim_{n\to\infty}\frac{1}{n+1} + \cdots + \lim_{n\to\infty}\frac{1}{2n} = 0 + \cdots + 0 = 0$$

   which is definitely wrong.
   When using the limit theorem

$$\lim(a_n + b_n + \cdots + z_n) = \lim a_n + \lim b_n + \cdots + \lim z_n$$

   the number of sequences involved is **finite and fixed**. But in our question the number of terms increases to infinity.
   The same mistake appears in Supplementary Exercise 2 in Assignment 5.

   (b) **3.4.7(d)** Let $e_n = \left(1 + \frac{1}{n}\right)^n$. It is known that $(e_n) \to e$ by definition. But it is illegal to derive

$$\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = \lim_{n\to\infty}\left[\left(1 + \frac{2}{n}\right)^{\frac{n}{2}}\right]^2 = e^2$$

   by regarding $\left(1 + \frac{1}{n/2}\right)^{\frac{n}{2}} \to e$ as a subsequence of (e_n) (In fact it is NOT a subsequence of (e_n)).

Part III: other problems.

1. **(Ex 3.5.6)** Let p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies $\lim |x_{n+p} - x_n| = 0$.

   Ans: Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be the example in **3.3.3 (b)**. Then (x_n) is not a sequence since it is divergent (you can also refer to the proof of divergence on page 73). However,

$$0 < |x_{n+p} - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{n+p} < \frac{p}{n+1}$$

   and by Squeeze Theorem we have $\lim_{n\to\infty} |x_{n+p} - x_n| = 0$.

2. **(Ex 3.4.16)** Given an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.

   Solutions: Consider the sequence

$$(x_n) = \left(1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \cdots\right).$$

   It can be checked that every convergent subsequence of (x_n) converges to $x = 0$ ((x_n) has only one limit point 0). However, (x_n) does not converge to 0.

3. (**Generalizations of Ex 3.4.19**) If (x_n) and (y_n) are bounded sequences, show that

   (a) $\varliminf_{n\to\infty} x_n + \varliminf_{n\to\infty} y_n \le \varliminf_{n\to\infty} (x_n + y_n) \le \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$;

   (b) $\varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n \le \varlimsup_{n\to\infty} (x_n + y_n) \le \varlimsup_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$.

   For each relation above, give an example for which strict inequality holds.

   Solution:

   (a) 1°. From **Question 4(c) in Part I**, there exists a subsequence $(x_{n_k} + y_{n_k})$ of $(x_n + y_n)$ such that $(x_{n_k} + y_{n_k}) \to \varliminf_{n\to\infty} (x_n + y_n) := c$.

   Now for the subsequence (x_{n_k}), there exists a subsubsequence $(x_{n_{k_i}}) \to \varliminf_{n\to\infty} x_{n_k} := a'$.

   Since

$$(y_{n_{k_i}}) = (x_{n_{k_i}} + y_{n_{k_i}}) - (x_{n_{k_i}}) \to c - a', \quad \text{(why?)}$$

   $c - a'$ is a limit point of (y_n) and thus

$$c - a' \ge \varliminf_{n\to\infty} y_n.$$

   Moreover, $a' = \varliminf_{n\to\infty} x_{n_k} \ge \varliminf_{n\to\infty} x_n$ (think about why) and therefore,

$$\varliminf_{n\to\infty} (x_n + y_n) - \varliminf_{n\to\infty} x_n \ge c - a' \ge \varliminf_{n\to\infty} y_n \implies \varliminf_{n\to\infty} x_n + \varliminf_{n\to\infty} y_n \le \varliminf_{n\to\infty} (x_n + y_n).$$

   2°. Similarly, there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k}) \to \varliminf_{n\to\infty} x_n := a$.

   Now for the subsequence (y_{n_k}), there exists a subsubsequence $(y_{n_{k_i}})$ such that

$$(y_{n_{k_i}}) \to \varlimsup_{n\to\infty} y_{n_k} := B'$$

   Since

$$(x_{n_{k_j}} + y_{n_{k_j}}) \to a + B', \quad \text{(why?)}$$

   $a + B'$ is a limit point of $(x_n + y_n)$ and thus

$$a + B' \ge \varliminf_{n\to\infty} (x_n + y_n).$$

   Moreover, $B' = \varlimsup_{n\to\infty} y_{n_k} \le \varlimsup_{n\to\infty} y_n$ (think about why) and therefore,

$$\varliminf_{n\to\infty} (x_n + y_n) \le a + B' = \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_{n_k} \le \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n.$$

3°. For examples, consider

$$\begin{aligned}(x_n) &= (1,0,1,0,1,0,\cdots),\\ (y_n) &= (0,2,0,2,0,2,\cdots),\\ (x_n+y_n) &= (1,2,1,2,1,2,\cdots).\end{aligned}$$

Then

$$\underline{\lim}_{n\to\infty} x_n = 0,\ \underline{\lim}_{n\to\infty} y_n = 0,\ \underline{\lim}_{n\to\infty} (x_n+y_n) = 1,\ \overline{\lim}_{n\to\infty} y_n = 2.$$

(b) Similar to (a), complete it yourself.

4. **(Question 2 on Feb 7 continued)**.

(a) Show that the conclusion in Question 2(a) on Feb 7 still holds if (x_n) is properly divergent, i.e.,

$$\lim_{n\to\infty} x_n = +\infty \Longrightarrow \lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = +\infty.$$

(b) Suppose (x_n) is a sequence of positive real numbers which converges to x. Show that

$$\lim_{n\to\infty} \sqrt[n]{x_1x_2\cdots x_n} = x.$$

(Hint: use the natural logarithm)

(c) **(Question 3 on Feb 7 continued: relation between ratio test and root test)** Suppose (x_n) is a sequence of positive real numbers. Show that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = x \Longrightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = x.$$

(d) Use (c) to find the limit $\lim_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}}$.

(e) Show that the converse of (c) is false by giving a counterexample.

(f) Suppose $(x_n), (y_n)$ are two sequences of real numbers and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Define a new sequence (z_n) by

$$z_n = \frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n}.$$

Show that (z_n) is also convergent and

$$\lim_{n\to\infty} z_n = xy.$$

Solution:

(a) Similar to Question 2(a) on Feb 7. Since $\lim_{n\to\infty} x_n = +\infty \Longrightarrow \forall M > 0$ there exists $N_1 \in \mathbb{N}$ such that $x_n \geq 3M$ whenever $n \geq N_1$.
By Archimedean Property, there exists $N_2 \in \mathbb{N}$ such that

$$\frac{|S_{N_1}|}{n} = \frac{|x_1+x_2+\cdots+x_{N_1}|}{n} \leq \frac{M}{2},\quad \frac{n-N_1}{n} > \frac{1}{2} \quad \text{whenever} \quad n \geq N_2.$$

Then for any $n \geq N := \max(N_1, N_2)$, we have

$$\begin{aligned}
|A_n| &= \frac{|x_1 + x_2 + \cdots + x_n|}{n} = \frac{|x_1 + x_2 + \cdots + x_{N_1} + x_{N_1+1} + \cdots + x_n|}{n} \\
&\geq \frac{|x_{N_1+1} + \cdots + x_n| - |x_1 + x_2 + \cdots + x_{N_1}|}{n} \\
&\geq \frac{3M(n - N_1) - |x_1 + x_2 + \cdots + x_{N_1}|}{n} \\
&\geq 3M \cdot \frac{n - N_1}{n} - \frac{|x_1 + x_2 + \cdots + x_{N_1}|}{n} \\
&\geq 3M \cdot \frac{1}{2} - \frac{M}{2} = M.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} A_n = +\infty$, i.e., A_n is also properly divergent.

(b) It can be seen that $x \geq 0$.

1°. If $x > 0$, then $\lim_{n\to\infty} \ln x_n = \ln x$ and we can use the result of Question 2(a) on Feb 7 to obtain

$$\lim_{n\to\infty} \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} = \ln x$$

and

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} e^{\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}} = e^{\ln x} = x.$$

2°. If $x = 0$, then $\lim_{n\to\infty} (-\ln x_n) = +\infty$ and by (a) we have

$$\lim_{n\to\infty} \frac{-\ln x_1 - \ln x_2 - \cdots - \ln x_n}{n} = +\infty$$

and hence

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} e^{-\frac{-\ln x_1 - \ln x_2 - \cdots - \ln x_n}{n}} = 0.$$

(c) Let $y_n = \dfrac{x_{n+1}}{x_n}, n = 1, 2, 3, \cdots$. Then (y_n) is a sequence of positive real numbers which converges to x. From (b) we have

$$\begin{aligned}
&\lim_{n\to\infty} \sqrt[n]{y_1 y_2 \cdots y_n} = x, \\
\Longrightarrow &\lim_{n\to\infty} \sqrt[n]{\frac{x_2}{x_1}\frac{x_3}{x_2} \cdots \frac{x_{n+1}}{x_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{x_{n+1}}{x_1}} = \lim_{n\to\infty} \frac{\sqrt[n]{x_{n+1}}}{\sqrt[n]{x_n}} = x.
\end{aligned}$$

It is a known result that $\lim_{n\to\infty} \sqrt[n]{x_1} = 1$ and we conclude that

$$\lim_{n\to\infty} \sqrt[n]{x_{n+1}} = x \Longrightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = x.$$

(d) Notice that $\dfrac{n}{\sqrt[n]{n!}} = \sqrt[n]{x_n}$ where $x_n = \dfrac{n^n}{n!}$. Since

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = e$$

we conclude from (c) that $\lim_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$.

(e) Consider the sequence

$$(x_n) = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^3}, \cdots\right).$$

Then we have $\lim_{n\to\infty} \sqrt[n]{x_n} = \frac{1}{\sqrt{2}} < 1$ (check it yourself) and it follows that (x_n) converges to 0. But **ratio test** fails because

$$\left(\frac{x_{n+1}}{x_n}\right) = \left(1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \cdots\right)$$

is divergent.

Remark: This question indicates that **root test** is stronger than **ratio test**.

(f) z_n can be rewritten as

$$\frac{(x_1 - x)y_n + (x_2 - x)y_{n-1} + \cdots + (x_n - x)y_1}{n} + x \cdot \frac{y_n + y_{n-1} + \cdots + y_1}{n} := S_1 + S_2.$$

(y_n) is convergent $\Rightarrow$ (y_n) is bounded by some positive number M. Therefore,

$$\begin{aligned} |S_1| &\leq \frac{|x_1 - x||y_n| + |x_2 - x||y_{n-1}| + \cdots + |x_n - x||y_1|}{n} \\ &\leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|}{n} M. \end{aligned}$$

Now $(x_n) \to x \Longrightarrow (x_n - x) \to 0 \Longrightarrow (|x_n - x|) \to 0$ and hence

$$\lim_{n\to\infty} \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|}{n} = 0 \Longrightarrow \lim_{n\to\infty} S_1 = 0.$$

Therefore,

$$\lim_{n\to\infty} z_n = 0 + x \cdot \lim_{n\to\infty} \frac{y_n + y_{n-1} + \cdots + y_1}{n} = 0 + x \cdot y = xy.$$

Part IV: On the mid-term examination

The topics we have studied:

- Real number system
  - The algebraic and order properties of $\mathbb{R}$
  - The Completeness Property of $\mathbb{R}$
- Limit of a sequence
  - Sequence and its limit
  - Limit theorems
  - Monotone Convergence Theorem

  - Subsequence and Bolzano-Weierstrass Theorem
  - Cauchy sequence

You should not expect the mid-term exam to be so easy as previous quizzes. Here are my suggestions:

- **Familiarize** yourself with all the theorems in the textbook and Prof's lecture notes, especially those bearing a name.
- Make sure that you have done **ALL** the exercises in the textbook yourself.